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SOLUTION OF ONE CLASS OF MAGNETOHYDRODYNAMIC EQUATIONS WITH MAGNETIC FIELD AMPLIFICATION

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ABSTRACT

The object of this paper is to work out a kind of dynamo-theory that could come close to model oscillating hydrodynamic dynamos of interest, among other applications, to solar physics.

To that effect, the author investigates to what magnetic configurations does the "hydromagnetic activity" of a convective cell lead if and when Coriolis forces are in action.

The author resolves the problem for the simplest model. Its interest resides in the fact that it may be easily extended from this simplest particular case to diversified situations with resembling velocity fields. These may in their turn be interesting for applications, and this is why they are discussed in the present work.

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Numerous observation data of the Sun suggest that there is below the solar photosphere surface a large-scale ("general") toroidal magnetic field, changing its sign every 11 years. For that reason model oscillating hydromagnetic dynamos are of interest to solar physics. The most characteristic type of plasma motion in the outer part of the Sun is the cellular Benarov-type convection (whereby the matter rises at the center of each cell and then flows sideways and descends at the edges). The deeper layers are convectively stable; one account of insignificant ohmic dissipation, there the magnetic field can hardly rebuild itself within a perceptible time lapse. It is natural to expect, that the convection in the sub-photospheric zone is to a great measure responsible for the operation of the solar dynamo. In the most general traits its possible scheme can be represented as follows.

The differential rotation of the Sun forms from the present weak meridional field a stronger azimuthal toroidal field. Each convective cell amplifies it still more by winding the lines of force of the toroidal field, while the action of the Coriolis forces results in the rotation of the entire pattern around the cell's axis. The diffusion of the magnetic field gradually smoothes out the inhomogeneities; owing to which a large-scale meridional field is generated with a sign opposite to the old one. After some time the meridional component of the general field changes its sign, and the whole process repeats itself.

To work out such a kind of dynamo-theory one must investigate to what magnetic configurations does the hydromagnetic "activity" of the convective cell lead, if and when the Coriolis forces are in action. In the proposed work this problem is solved for the simplest model. It is interesting that from the chosen particular case, one can easily pass to diversified situations with resembling velocity fields. They may also be interesting in applications and that is why they are being discussed in the present work.

In connection with the dynamo-problem, Tverskoy [1] considered an axisymmetric vortex ring, in which the velocity of matter had no azimuthal component. The current lines are disposed in meridional vortex cross-section planes and constitute circumferences with centers at a distance $\underline{a}$ from the axis. All lines of radius $\underline{r}$ from the surface of a circular tore. It is then practical to make use of the orthogonal system of coordinates r, ϕ, χ, where ϕ is the azimuthal angle and χ is the polar angle in the plane of the meridional cross-section. The Lamé parameters are $h_r = 1$, $h_\phi = a + r \cos \chi$, $h_\chi = r$.

Having assumed that not only v_χ, but also v_ϕ are not zero, we shall obtain a schematic representation of a convective cell with Coriolis perturbation of the velocity field. From the condition of incompressibility $\operatorname{div} \vec{v} = 0$, we find

$$v_\chi = V(r)/[1 + (r/a)\cos \chi],$$

where $V(r)$ is an arbitrary function equal to zero for $r \geqslant r_0$ $(r_0 < a)$. We shall consider that v_ϕ varies as a linear function of distance from the vertical axis $a + r \cos \chi$ along each χ-line, i.e.

$$v_\phi = U[b(r) - r \cos \chi],$$

U being a constant dependent on the rotation velocity of the fluid as a whole. If the magnetic field is sufficiently weak, one may neglect its decelerating influence on the plasma and consider the motion as stationary (problem of kinematics). At great magnetic Reynolds numbers and in the course of a certain lapse of time beginning

with the time of motion onset in the cell, the magnetic field is subject to the law of induction for an infinetly conducting fluid

$$\partial\vec{H}/\partial t = \mathrm{rot}[\vec{v}\vec{H}]$$

or in components

$$\frac{\partial H_r}{\partial t} = -\frac{1}{r(a + r\cos\chi)}\{r v_\phi(r,\chi)\frac{\partial H_r}{\partial\phi} + aV(r)\frac{\partial H_r}{\partial\chi}\} \quad (1)$$

$$\frac{\partial H_\phi}{\partial t} = \frac{1}{r}\frac{\partial}{\partial\chi}|v_\phi(r,\chi)H_\chi - \frac{aV(r)}{a + r\cos\chi}H_\phi| + \frac{\partial}{\partial r}[r v_\phi(r,\chi)H_r]\}; \quad (2)$$

$$\frac{\partial H_\chi}{\partial t} = \frac{1}{a + r\cos\chi}\{a\frac{\partial}{\partial r}[V(r)H_r] + \frac{aV(r)}{a + r\cos\chi}\frac{\partial H_\phi}{\partial\phi} - v_\phi(r,\chi)\frac{\partial H_\chi}{\partial\phi}\} \quad (3)$$

We shall seek the solution of Eq. (1) in the form

$$H_r = f(r,\chi)e^{i(\omega t + m\varphi)}.$$

Having determined the function $f(r,\chi)$ and applying it the condition of periodicity with respect to χ, we shall obtain the discrete spectrum of frequencies.

$$\omega_{mn}(r) = n\Omega(r) - m\Psi(r). \quad (4)$$

Here $\Omega(r) = V(r)/r$ (frequency of liquid particle motion along χ), $\Psi(r) = Ub(r)/a$ (mean velocity of its displacement along ϕ); m, n = = $0 \pm 1, \pm 2, \ldots$. Finally

$$H_r^{mn} = A_r^{mn}(r)f_{mn}(r,\chi)e^{i[\omega_{mn}(r)t + m\varphi]}, \quad (5)$$

where

$$f_{mn} = \exp\{-in[\chi + (r/a)\sin\chi] + imU(r/a\Omega)(b/a + 1)\sin\chi\},$$

and $A_r^{mn}(r)$ is an arbitrary function.

Taking into account that div $\vec{H} = 0$, we shall eliminate H_ϕ from (3) and thus obtain a line equation for H_χ with right-hand part proportional to H_r. Substituting (5) it is possible to find the partial solutions H_χ^{mn}. But if we consider that H_χ^{mn} depends on <u>t</u> as it will represent a traveling wave along χ and ϕ, of which the amplitude increases infinitely with χ. The periodical solution with respect to χ must include instead of spatial rise, the **temporal** one,

and have the form

$$H_\chi^{mn} = [\tilde{A}_\chi^{mn}(r,\chi) + \tilde{B}_\chi^{mn}(r,\chi)t] f_{mn} e^{i[\omega_{mn}(r)t + m\varphi]}. \tag{6}$$

Then we obtain for $\tilde{B}_\chi^{mn}$ and $\tilde{A}_\chi^{mn}$ differential equations, of which one yields

$$\tilde{B}_\chi^{mn} = B_\chi^{mn}(r)/(a + r\cos\chi)$$

B_χ^{mn} being an arbitrary function. From the second equation, upon substitution in it $\tilde{B}_\chi^{mn}$, we find $\tilde{A}_\chi^{mn}$ dependent on $\tilde{B}_\chi^{mn}(r)$ and on the arbitrary function $A_\chi^{mn}(r)$. Having applied to $\tilde{A}_\chi^{mn}(r,\chi)$ the requirement of periodicity, we determine $B_\chi^{mn}(r)$. As a result

$$H_\chi^{mn} = \frac{1}{a + r\cos\chi}\left[A_\chi^{mn}(r) - r\sin\chi A_r^{mn}(r) + ar\frac{d\Omega}{dr}A_r^{mn}(r)t\right] f_{mn} e^{i(\omega_{mn}t + m\varphi)}. \tag{7}$$

Finally from Eq.2, which incidently is also easy to transform with the aid of equality div $\vec{H} = 0$, we find H_ϕ^{mn} entirely analogously upon substitution of (5) and (7). The expression for $\tilde{A}_\phi^{mn}$ is cumbersome. Here it makes sense to write out only $\tilde{B}_\phi^{mn}$, which determines the part of H_ϕ^{mn} growing in time:

$$B_\varphi^{mn} = U\left\{\frac{db}{dr}\left(1 + \frac{r}{a}\cos\chi\right) - \frac{1}{\Omega}\frac{d\Omega}{dr}[a + b(r)]\frac{r}{a}\cos\chi\right\} A_r^{mn}(r). \tag{8}$$

The system of functions $e^{-in[\chi + (r/a)\sin\chi]}$ is orthogonal relative to weight $1 + (r/a)\cos\chi$ and complete, inasmuch as it is reduced to system e^{-inz} by the mutually single-valued relation between z and χ. Any function of r, ϕ and χ may be represented by combination $f_{mn}e^{im\varphi}$. To that effect it must be expanded by $e^{im\varphi}$, the coefficients of expansion multiplied by $\exp\{-imU(r/a\Omega)(b/a + 1)\sin\chi\}$ and expanded by e^{-inz}. In this way A_r^{mn}, A_ϕ^{mn} and A_χ^{mn} can, in principle, be always found from the initial conditions.

The multiplier $f_{mn}(r,\chi)e^{i(\omega_{mn}t + m\varphi)}$ describes the transfer of the magnetic field along ϕ and χ. Because of flow inhomogeneity the spatial distribution of all harmonics H^{mn} with $n \neq 0$ becomes more complex with time, for the periods of functions $e^{i\omega_{mn}(r)t}$ become shorter along r. This inhomogeneity leads to stretching of the field lines and to the increases of amplitudes of H_ϕ^{mn} and H_χ^{mn}.

The rapidity of the increase depends on gradients of mean angular velocities Ω and Ψ. The terms proportional to t may become prevailing over all the others, including H_r^{mn} for a sufficient time. The growing part of harmonic H^{00} is characteristic in that its distribution in space does not vary. If $v_\phi << v_\chi$, the lower nonzero harmonics with respect to m with n = 0 maintain their regularity in space, slowly shifting along ϕ.

As was shown by Tverskoy [1] the regular harmonics are those namely essential at computation of a large-scale field generated by a convective zone. This is why in order to investigate the operation of the dynamo it is sufficient to separate in the found solution the terms of time-growing zero harmonics with respect to $\underline{n}$. If the initial field is uniform and horizontal with the bounds of the cell; only components with m = 1 are excited, and $A_r^{10}(r)$ is expressed by a Bessel function.

The described scheme of the solution is also valid for a series of other problems. Assume that in an orthogonal system of coordinates q_1, q_2, q_3 with Lamé parameters h_1, h_2, h_3, the fluid's velocity vector has the form $\vec{v} = \{v_1, v_2, 0\}$ and that all the current lines lie on surfaces q_3 = const. Let us introduce the "normalized" components of vectors $\vec{H}$ and $\vec{v}$:

$$B_i = H_i/h_i, \quad u_i = v_i/h_i \quad (i = 1,\ 2,\ 3;\ u_3 = 0)$$

and denote $h_1 h_2 h_3 = h$. Let, moreover, u_1 and u_2 be independent of q_1 (or q_2; for determination we assume the first one). Besides the functions

$$\Phi(q_2, q_3) = \int_0^{q_2} u_2^{-1}\, dq_2, \quad X(q_2, q_3) = \int_0^{q_2} (u_1/u_2)\, dq_2,$$

are defined everywhere in the region of flow, whereupon u_2 is never zero. The equation div $\vec{H}$ = 0 and div $\vec{v}$ = 0 allow us to write the law of induction as in Descartes coordinates:

$$\partial B_1 / \partial t = B_2 \partial u_1 / \partial q_2 + B_3 \partial u_1 / \partial q_3 - u_1 \partial B_1 / \partial q_1 - u_2 \partial B_1 / \partial q_2; \tag{9}$$

$$\partial B_2 / \partial t = B_2 \partial u_2 / \partial q_2 + B_3 \partial u_2 / \partial q_3 - u_1 \partial B_2 / \partial q_1 - u_2 \partial B_2 / \partial q_2; \tag{10}$$

$$\partial B_3 / \partial t = -u_1 \partial B_3 / \partial q_1 - u_2 \partial B_3 / \partial q_2. \tag{11}$$

One may always select on the plane an orthogonal net of coordinates q_2, q_3, that would include the prescribed family of smooth vectorial lines of vector $\vec{v} = \{0, v_2, 0\}$. This is why the introduced requirements are fulfilled and the further written solution is valid, so long as the flow is axisymmetrical (q_1 being the azimuthal angle) or when the q_1-lines are parallel straight lines, while the velocity field is symmetrical relative to the transfer along these lines. Related to such a type is, for example, the outflow of solar wind: superimposed on the motion of radial straight q_2-lines is the rotation around the axis of symmetry with angular velocity dropping as the range from the center increases. Possible more comples velocity

fields satisfy the enumerated conditions: for these it is useful to resolve the problem of the magnetic field.

System (9)-(11) is resolved in the same manner as (1)-(3). The particular solution depending on arbitrary k and ω is as follows:

$$B_1^{k\omega} = \{ A_1^{k\omega} + A_2^{k\omega} u_1 + A_3^{k\omega} (\partial X/\partial q_3 - u_1 \partial\Phi/\partial q_3) + (C_1^{k\omega} + C_2^{k\omega} u_1)(t - \Phi) \} \varphi^{k\omega}; \quad (12)$$

$$B_2^{k\omega} = u_2 \{ A_2^{k\omega} - A_3^{k\omega} \partial\Phi/\partial q_3 + C_2^{k\omega} (t - \Phi) \} \varphi^{k\omega}; \quad (13)$$

$$B_3^{k\omega} = A_3^{k\omega} \varphi^{k\omega}. \quad (14)$$

Here all $A_i^{k\omega}$ and $C_i^{k\omega}$ are arbitrary function of q_3, and $\phi^{k\omega} = e^{i\omega(t-\Phi)+ik(q_1-X)}$. Depending upon the statement of the problem, the solution, satisfying the initial condition may be represented either by series of functions (12)-(14) or by their integrals over k and ω. For example, in the problem of solar wind the region of flow is infinite and ω has a continuous spectrum of values. At the same time one should postulate $C_1^{k\omega} = 0$ (a nonperiodical dependence on t may also be represented by a Fourier integral). But if the q_2-lines are closed (as in the case of vortex) a discrete multitude of values of ω will correspond to the given k; the periodicity with respect to q_2 also dictates a specific choice of $C_1^{k\omega}$. Formulas (12)-(14) are easiest to use when q_1-lines are closed. If q_1 and q_2 vary within the limits from 0 to 2π, ω_{mn} will be given by formula (4), where one should postulate:

$$\Omega(q_3) = 2\pi / \Phi(2\pi, q_3), \quad \Psi(q_3) = X(2\pi, q_3) / \Phi(2\pi, q_3). \quad (15)$$

Then

$$C_1^{mn} = A_3^{mn} [d\Psi/dq_3 - (\Psi/\Omega) d\Omega/dq_3], \quad C_2^{mn} = A_3^{mn} \Omega^{-1} d\Omega/dq_3, \quad (16)$$

where

$$\varphi^{mn} = \exp\{ i\omega_{mn} (t - \Phi) + im(q_1 - X) \}.$$

If B_{0i} are the initial values of components of vector $\vec{B}$, then $A_j^{mn}(q_3)$ is determined as the coefficients of expansions

$$F_{jm} e^{im(X - \Psi\Phi)} = \sum_n A_j^{mn} e^{-in\Omega\Phi}$$

by a system of functions of variable q_2, which are orthogonal relative

the weight $\partial(\Omega\Phi)/\partial q_2$, whereupon $F_{jm}(q_2, q_3)$ are the amplitudes in expansions

$$B_{01} - B_{02}u_1 + [(u_1 - \Psi)(\Phi/\Omega)\, d\Omega/dq_3 + \Phi\, d\Psi/dq_3 + $$
$$+ u_1\, \partial\Phi/\partial q_3 - \partial X/\partial q_3]\, B_{03} = \sum_m F_{1m} e^{imq_1},$$
$$B_{02}/u_2 + [\partial\Phi/\partial q_3 + (\Phi/\Omega)\, d\Omega/dq_3]\, B_{03} = \sum_m F_{2m} e^{imq_1},$$
$$B_{03} = \sum_m F_{3m} e^{imq_1}.$$

The fundamental traits of the solution of the generalized problem are discerned from analogy with the case of toroidal vortex. If $v_1 = 0$, (12)-(14) pass into formulas of the work [2], which are also valid at $\partial u_2/\partial q_1 \neq 0$.

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R E F E R E N C E S

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